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ON THE OPTIMUM SHAPE OF A ROTATING DISK OF ANY ISOTROPIC MATERIAL

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Abstract—A new approach is proposed to the solution of the optimum shape of a rotating disk. This is based principally on the new starting assumption (Section 2), on the approximations in the principal stresses and stress trajectories (Section 3), and on the symmetry properties of the strain–stress relations due to isotropy, as well as on the Maclaurin series expansion of these relations with respect to σ_z and τ_{rz} (Section 4). The optimum shape deduced is the classical one which diminishes exponentially (Section 5), but here the relative error can be estimated (Section 6).

1. INTRODUCTION

IN THIS context, the concept "optimum" implies that the disk has a uniform strength throughout. To minimize the weight of the disk, its material will be exploited in extreme strength, and nowhere must there be excess material. The last requirement is fulfilled when the strength is uniform. The applicable criteria of the uniformity are specified in Section 5 below. The ultimate allowable stress, in turn, is decided by the safety. The disk must be designed for the fixed operating speed of rotation, which is often very high, and thus any possible damage would be extremely great. The optimum shape is an important aspect of construction.

Two different approaches have been made to the problem. In the simpler approach, the solution has been derived merely on the basis of equilibrium considerations, after the *a priori* assumption that the radial and tangential stresses are both equal to the same constant σ, $\sigma_r = \sigma = \sigma_\theta$. As the number of variables is greater by one than that of the equations, in a more sophisticated approach the constancy of only the tangential stress, $\sigma_\theta = \sigma$, has been assumed as an additional condition, and the generalized Hooke's law is applied to prove that $\sigma_r = \sigma$. In both cases, the treatment is uncertainly approximate, since the stresses σ_z and τ_{rz} are ignored, and no estimate of error is provided. The design obtained is a disk with exponentially diminishing thickness.

Neither of these approaches is employed expressly in this paper. The preference is given to an additional condition such that the ratio σ_θ/σ_r is a constant. The strain–stress relation is not assumed to be linear, but is determined by any analytic law which fulfils some natural requirements attributable to isotropy. Within the accuracy of the error estimations given, it is then proved that $\sigma_r = \sigma = \sigma_\theta$. In spite of the essential generalizations made in regard to the elastic behaviour of the material, the resulting shape of the optimal disk is the classical exponential shape.

In the formula there appears the material parameter, ρ/σ, the density of the material divided by the working stress. As a reduction in this parameter will lighten the disk, the natural tendency is towards lighter materials. Since such materials often have a non-linear stress–strain relation, the applicability of the classical results is doubtful. In this non-linear case, the engineering approach to the problem may not present so many difficulties as may have been believed.

Cylindrical coordinates and the conventional symbols are used in what follows.

2. EQUILIBRIUM EQUATIONS

If the angular speed is ω and the density of material is ρ, then the centrifugal force per unit volume is $\rho\omega^2 r$ where r is the distance from the axis of rotation. It is evident that the configuration possesses the rotational symmetry about the axis of rotation, say the z-axis, and that there exists a plane of mirror symmetry where $z = 0$. Consequently, there is no dependence upon the angular coordinate θ, and thus the equations of equilibrium read (see [1] p. 352 equation 189)

$$\begin{aligned}\frac{\partial \sigma_r}{\partial r}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}+\rho\omega^2 r &= 0\\ \frac{\partial \tau_{rz}}{\partial r}+\frac{\partial \sigma_z}{\partial z}+\frac{\tau_{rz}}{r} &= 0.\end{aligned} \tag{1}$$

Let the surfaces of the disk be given by

$$z = \pm h(r)/2. \tag{2}$$

Since these surfaces are unloaded, the boundary conditions are

$$\left.\begin{aligned}\pm(h'/2)\sigma_r-\tau_{rz} &= 0\\ \pm(h'/2)\tau_{rz}-\sigma_z &= 0\end{aligned}\right\}\text{for } z = \pm h/2, \tag{3}$$

in which the prime denotes the differentiation with respect to r. The equations (1) are now integrated with respect to z from $-h/2$ to $+h/2$, and abbreviations are introduced:

$$T = \int_{-h/2}^{+h/2} \tau_{rz}\,\mathrm{d}z \tag{4}$$

$$S_r = \int_{-h/2}^{+h/2} \sigma_r\,\mathrm{d}z \tag{5}$$

$$S_\theta = \int_{-h/2}^{+h/2} \sigma_\theta\,\mathrm{d}z \tag{6}$$

and equations (3) are applied to obtain

$$\frac{\partial}{\partial r}(rS_r)-S_\theta+\rho h\omega^2 r^2 = 0 \tag{7}$$

$$\frac{\partial}{\partial r}(rT) = 0. \tag{8}$$

The last equation can be integrated immediately

$$T = A/r \quad \text{where } A = \text{constant}. \tag{9}$$

This has a singularity at $r = 0$ unless $A \equiv 0$, which leads in turn to $T \equiv 0$. This is possible only if τ_{rz} is an odd function of z

$$\tau_{rz}(z) = -\tau_{rz}(-z) \tag{10}$$

or identically zero. In reality, this is a direct consequence of the mirror symmetry with respect to the plane $z = 0$.

For the further treatment of equation (7), average stresses are defined by the formulae

$$S_r = \bar{\sigma}_r h \quad \text{and} \quad S_\theta = \bar{\sigma}_\theta h. \tag{11}$$

After substitution into equation (7), there is obtained

$$\frac{\partial}{\partial r}(rh\bar{\sigma}_r) - h\bar{\sigma}_\theta + h\rho\omega^2 r^2 = 0. \tag{12}$$

Since there are more dependent variables than equations, an additional condition can be introduced. For instance, it can be demanded that the disk is such that the ratio σ_θ/σ_r is a constant, say s

$$s = \sigma_\theta/\sigma_r, \quad \text{and thus} \quad s = \bar{\sigma}_\theta/\bar{\sigma}_r. \tag{13}$$

The equation (12) can now be rewritten

$$\frac{\partial}{\partial r}(rh\bar{\sigma}_r) - (rh\bar{\sigma}_r)s/r + h\rho\omega^2 r^2 = 0. \tag{14}$$

This is a differential equation of $(rh\bar{\sigma}_r)$, the integration of which yields

$$\Big/_0^r \ln(r^{1-s}h\bar{\sigma}_r) = -\rho\omega^2 \int_0^r \frac{r\,\mathrm{d}r}{\bar{\sigma}_r}. \tag{15}$$

If the existence of the realistic limit $h \to h_0 =$ constant is implied as $r \to 0$, the ln-function then has a branch point at $r = 0$, unless $s \equiv 1$, which means

$$\begin{aligned} \sigma_r &= \sigma_\theta \quad \text{and thus} \\ \bar{\sigma}_r &= \bar{\sigma}_\theta. \end{aligned} \tag{16}$$

Consequently, the solution of equation (14) reads

$$\frac{h\bar{\sigma}_r}{(h\bar{\sigma}_r)_{r=0}} = \exp\left(-\rho\omega^2 \int_0^r \frac{r\,\mathrm{d}r}{\bar{\sigma}_r}\right). \tag{17}$$

3. PRINCIPAL STRESSES AND STRESS TRAJECTORIES

We define now a stress matrix

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_r & 0 & \tau_{rz} \\ 0 & \sigma_\theta & 0 \\ \tau_{rz} & 0 & \sigma_z \end{pmatrix} \tag{18}$$

and a strain matrix

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \dfrac{\partial u}{\partial r} & 0 & \dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial r} \\ 0 & \dfrac{u}{r} & 0 \\ \dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial r} & 0 & \dfrac{\partial w}{\partial z} \end{pmatrix} \tag{19}$$

Reference should be made to [1] p. 343 Formulae (177) and (178) for their components. u is the radial displacement, and w is the displacement in the z-direction. To arrive at the principal stresses ([1] p. 217 or [2] p. 47), there is formed the characteristic equation of matrix $\boldsymbol{\sigma}$ (cf. [5] p. 66 and p. 76)

$$\Delta(\lambda) = |\lambda \mathbf{I} - \boldsymbol{\sigma}| = 0 \tag{20}$$

where $\mathbf{I}$ is the identity matrix, λ is an eigenvalue or characteristic root, and $\Delta(\lambda)$ is a characteristic polynomial equal to the expanded form of the determinant $|\lambda \mathbf{I} - \boldsymbol{\sigma}|$. By virtue of equation (18), this determinant can be written

$$\begin{vmatrix} \lambda-\sigma_r & 0 & -\tau_{rz} \\ 0 & \lambda-\sigma_\theta & 0 \\ -\tau_{rz} & 0 & \lambda-\sigma_z \end{vmatrix} = 0. \tag{21}$$

Its expansion leads to

$$\Delta(\lambda) = (\lambda-\sigma_\theta)[\lambda^2-(\sigma_r+\sigma_z)\lambda+\sigma_r\sigma_z-\tau_{rz}^2] = 0$$

from which are solved the characteristic roots

$$\begin{aligned} \lambda_1 &= \sigma_\theta \\ \lambda_2 &= \tfrac{1}{2}(\sigma_r+\sigma_z)+\tfrac{1}{2}\sqrt{[(\sigma_r-\sigma_z)^2+4\tau_{rz}^2]} \\ \lambda_3 &= \tfrac{1}{2}(\sigma_r+\sigma_z)-\tfrac{1}{2}\sqrt{[(\sigma_r-\sigma_z)^2+4\tau_{rz}^2]}. \end{aligned} \tag{22}$$

Roots λ_2 and λ_3 satisfy the following well-known equations

$$\begin{aligned} \lambda_2+\lambda_3 &= \sigma_r+\sigma_z \\ \lambda_2\lambda_3 &= \sigma_r\sigma_z-\tau_{rz}^2. \end{aligned} \tag{23}$$

The expressions in (22) for λ_2 and λ_3 resemble those of the principal plane stresses ([1] p. 17 equation (16)). The equation for the principal directions or stress trajectories now reads ([1] p. 14 equation (14))

$$\tan 2\alpha = \frac{2\tau_{rz}}{\sigma_r-\sigma_z}$$

or in terms of a single slope $\tan\alpha = dz/dr$, a quadratic equation is derived:

$$\tan^2\alpha+\frac{\sigma_r-\sigma_z}{\tau_{rz}}\tan\alpha-1 = 0, \tag{24}$$

which has the roots

$$\tan\alpha_{2,3} = -\{\tfrac{1}{2}(\sigma_r-\sigma_z)\pm\tfrac{1}{2}\sqrt{[(\sigma_r-\sigma_z)^2+4\tau_{rz}^2]}\}/\tau_{rz}.$$

On the application of expressions (22) and (23), these roots are found to be:

$$\begin{aligned}\tan\alpha_2 &= -(\lambda_3-\sigma_z)/\tau_{rz} \equiv -(\sigma_r-\lambda_2)/\tau_{rz}\\ \tan\alpha_3 &= -(\lambda_2-\sigma_z)/\tau_{rz} \equiv -(\sigma_r-\lambda_3)/\tau_{rz}.\end{aligned} \tag{25}$$

It can be anticipated that the outer surface $z = \pm h/2$ is a stress trajectory. From the boundary conditions (3), there is obtained for the plus sign or $z = h/2$ only,

$$\begin{aligned}\tau_{rz} &= \frac{h'}{2}\sigma_r\\ \sigma_z &= \frac{h'}{2}\tau_{rz} = \left(\frac{h'}{2}\right)^2\sigma_r.\end{aligned} \tag{26}$$

These have the immediate consequence

$$\sigma_r\sigma_z-\tau_{rz}^2 = 0, \tag{27}$$

which results in

$$\begin{aligned}\lambda_2 &= \sigma_r+\sigma_z = \left(1+\frac{h'^2}{4}\right)\sigma_r, &\quad \tan\alpha_2 &= \frac{h'}{2}\\ \lambda_3 &= 0, & \tan\alpha_3 &= -\frac{2}{h'}.\end{aligned} \tag{28}$$

The surface $z = 0$ is a stress trajectory, by virtue of mirror symmetry. Since τ_{rz} was an odd function of z, equation (10), then $\tau_{rz} \equiv 0$ for $z = 0$, and consequently

$$\begin{aligned}\lambda_2 &= \sigma_r, &\quad \tan\alpha_2 &= 0\\ \lambda_3 &= \sigma_z, & \tan\alpha_3 &= -\infty.\end{aligned} \tag{29}$$

By means of equation (16) $\sigma_r = \sigma_\theta$, it is an easy matter to derive from equations (1)

$$\frac{\partial^2\sigma_z}{\partial z^2} = 2\rho\omega^2+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\sigma_r}{\partial r}\right).$$

If σ_r is at least nearly constant, as is desired, this expression is not negative, and then σ_z is obviously a non-negative monotonically increasing even function of z, possessing its maximum value at the boundaries $z = \pm h/2$. As a result, by virtue of equations (26), (28) and (29) it is certain that everywhere

$$\begin{aligned}1 &\le \frac{\lambda_2}{\sigma_r} \le 1+\tan^2\alpha_2\\ 0 &\le \frac{\lambda_3}{\sigma_r} \le \tan^2\alpha_2.\end{aligned} \tag{30}$$

If the thickness of the disk is varying slowly, as has been assumed, then inequalities (30) provide good reason to believe that for every r, and z in $-h/2 \le z \le +h/2$,

$$\begin{aligned}\frac{\lambda_2}{\sigma_r} &= 1\\ \frac{\lambda_3}{\sigma_r} &= 0.\end{aligned} \tag{31}$$

The error is of the order $O(\tan^2\alpha_2)$. This approximation has the evident consequence that every stress trajectory $z_2 = z_2(r)$ can be taken for an outer surface of the disk, and that they are all expressible in the form

$$\frac{z'}{z} = \zeta'(r) \quad \text{where } \zeta'(r) \equiv \frac{h'}{h}. \tag{32}$$

The expressions for τ_{rz} and σ_z are obtained if $h'/2$ is replaced by $z\zeta'(r)$ in equations (26)

$$\begin{aligned}\tau_{rz} &= z\zeta'(r)\sigma_r\\ \sigma_z &= [z\zeta'(r)]^2\sigma_r.\end{aligned} \tag{33}$$

Since $|z\zeta'(r)| \le |h'/2| = |\tan\alpha_2|$ they lead to the inequalities

$$\begin{aligned}|\tau_{rz}|/\sigma_r &\le |\tan\alpha_2|\\ \sigma_z/\sigma_r &\le \tan^2\alpha_2.\end{aligned} \tag{34}$$

4. STRAIN–STRESS RELATIONS

For an isotropic non-linear material, the strain–stress relations can be assumed to be

$$\begin{aligned}\frac{\partial u}{\partial r} &= f(\sigma_r, \sigma_\theta, \sigma_z) \equiv f(\sigma_r, \sigma_z, \sigma_\theta)\\ \frac{u}{r} &= f(\sigma_\theta, \sigma_z, \sigma_r) \equiv f(\sigma_\theta, \sigma_r, \sigma_z)\end{aligned} \tag{35}$$

$$\frac{\partial w}{\partial z} = f(\sigma_z, \sigma_r, \sigma_\theta) \equiv f(\sigma_z, \sigma_\theta, \sigma_r) \tag{36}$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = g(\tau_{rz}) \tag{37}$$

In the functions f, the last two variables can be interchanged by reason of isotropy. If the substitution σ_r for σ_θ in equations (35), is now introduced in accordance with the statement (16), then a differential equation in u is found immediately, viz.

$$\frac{\partial u}{\partial r} = \frac{u}{r}. \tag{38}$$

This has the only non-trivial solution

$$u = re(z), \tag{39}$$

where $e(z)$ is an arbitrary even function of z.

On the basis of the inequalities (34) and since the importance of the first variable σ_1 in $f(\sigma_1, \sigma_2, \sigma_3)$ is not less than that of σ_2 and σ_3, σ_z can be ignored on comparison with σ_r. In other words, σ_z is put equal to zero in equations (35) and (36). By application of the equations (35) and (39), there is obtained

$$f(\sigma_r, \sigma_r, 0) = e(z), \tag{40}$$

while the relative error is of the order $O(|f'_3(\sigma_r, \sigma_r, 0)\sigma_z/f(\sigma_r, \sigma_r, 0)|)$, where $f'_3 = (\partial/\partial\sigma_3)f(\sigma_1, \sigma_2, \sigma_3)$ indicates the derivative with respect to the third variable. If $e(z)$ is considered as known, this equation (40) defines, when solved in σ_r, a function

$$\sigma_r = \sigma_r(z) \equiv f^{-1}[e(z)]$$

σ_r is thus independent of r. From equation (36) there is derived

$$f(0, \sigma_r, \sigma_r) = \frac{\partial w}{\partial z}. \tag{41}$$

The relative error in this approximation is of the order $O(|f'_1(0, \sigma_r, \sigma_r)\sigma_z/f(0, \sigma_r, \sigma_r)|)$ where $f'_1 = (\partial/\partial\sigma_1)f(\sigma_1, \sigma_2, \sigma_3)$ indicates the derivative with respect to the first variable. Then, by equation $\sigma_r = \sigma_r(z)$, and after integration, there is obtained

$$w(z) = F(z) \equiv \int_0^z f[0, \sigma_r(z), \sigma_r(z)]\, dz \tag{42}$$

w is also independent of r. It is evident that the function g in equation (37) is an odd function. Consequently $g(0) = 0$, and since $\partial w/\partial r \equiv 0$ and $\partial u/\partial z = re'(z)$, by virtue of equation (33), there is derived from equation (37) the Maclaurin series expansion

$$re'(z) = g'(0)z\zeta'(r)\sigma_r. \tag{43}$$

Integration of this yields

$$e(z) = e + \tfrac{1}{2}g'(0)z^2\sigma_r\zeta'(r)/r \tag{44}$$

where e is a constant. If the approximation is made

$$e(z) = e, \tag{45}$$

the relative error made is of the order $O(|\tfrac{1}{2}g'(0)z^2\sigma_r\zeta'(r)/r|/f(\sigma_r, \sigma_r, 0))$. This approximation has the immediate consequence that the radial and tangential stresses are a constant σ throughout

$$\sigma_r = \sigma_\theta = \bar{\sigma}_\theta = \bar{\sigma}_r = \text{constant} = \sigma, \tag{46}$$

as are also the average stresses. Displacements u and w read

$$\begin{aligned} u &= rf(\sigma, \sigma, 0) = re \\ w &= zf(0, \sigma, \sigma). \end{aligned} \tag{47}$$

5. OPTIMUM SHAPE

Since $\bar{\sigma}_r$ is a constant σ, then from equation (17) there is obtained for the thickness

$$h = h_0 \exp(-\tfrac{1}{2}\rho\omega^2 r^2/\sigma). \tag{48}$$

This is exactly the same equation as that derived by application of the generalized Hooke's law (see [3] Part II p. 897 equation (g)). It should be pointed out that the expression (48) is independent of the stress–strain relation. The radial stress $\sigma_r = \sigma$ cannot vanish at the outer periphery but must be produced by the centrifugal force of external loads. It appears from equations (22), (31) and (46) that the principal stresses λ_1 and λ_2 are equal constants, while λ_3 vanishes

$$\begin{aligned} \lambda_1 &= \sigma_\theta = \sigma \\ \lambda_2 &= \sigma_r = \sigma \\ \lambda_3 &= 0 \end{aligned} \tag{49}$$

It is easily demonstrable by equations (19), (38) and (47) that the principal strains are

$$\begin{aligned} \varepsilon_1 &= \frac{u}{r} = f(\sigma, \sigma, 0) = e \\ \varepsilon_2 &= \frac{\partial u}{\partial r} = f(\sigma, \sigma, 0) = e \\ \varepsilon_3 &= \frac{\partial w}{\partial z} = f(0, \sigma, \sigma). \end{aligned} \tag{50}$$

In view of these considerations, it is clear that a disk of an exponential thickness, (equation (48)), has an optimum shape by reason of uniform strength, in accordance with maximum stress, strain, and shear hypotheses.

The function $\zeta'(r) = h'/h$ is calculated by equation (48)

$$\zeta'(r) = -\rho\omega^2 r/\sigma, \tag{51}$$

and the equations (33) now give

$$\begin{aligned} \tau_{rz} &= -\rho\omega^2 rz \\ \sigma_z &= \frac{(\rho\omega^2)^2}{\sigma} r^2 z^2. \end{aligned} \tag{52}$$

On direct substitution of these into the basic differential equations (1), it can be verified that τ_{rz} will satisfy the first one, although σ_z does not satisfy the second. Equations (1) are satisfied by a slightly modified σ_z, that is by the pair

$$\begin{aligned} \tau_{rz} &= -\rho\omega^2 rz \\ \sigma_z &= \rho\omega^2 z^2 \end{aligned} \tag{53}$$

σ_z has no other terms, since on the basis of equations (29), (30) and (31) it was approximated by zero for $z = 0$. This pair of τ_{rz} and σ_z does not fulfill, in turn, the requirement $\sigma_r\sigma_z - \tau_{rz}^2 = 0$ due to $\lambda_3 = 0$.

The error estimations contain the absolute value of the slope of a stress trajectory $|\tan \alpha_2|$. This assumes its maximum value on the outer surface at the circle (cf. [3] II p. 898)

$$r = \sqrt{\left(\frac{\sigma}{\rho\omega^2}\right)}, \tag{54}$$

where the contour has an inflection point. This maximum value is

$$\max |\tan \alpha_2| = \tfrac{1}{2}\sqrt{\left(\frac{\rho\omega^2 h_0^2}{\sigma}\right)} \exp(-\tfrac{1}{2}). \tag{55}$$

Since by virtue of equation (51) $\zeta'(r)/r = -\rho\omega^2/\sigma$, the relative error, on taking $e(z) = e =$ constant, can be inferred by equation (55) to amount to

$$O\left(\frac{\exp(1)}{2} g'(0)\sigma \max(\tan^2\alpha_2)/f(\sigma, \sigma, 0)\right) \tag{56}$$

In practical applications, the centrifugal load is known and after the choice of the allowable stress σ, the thickness $h(R)$ at the periphery $r = R$ is determined. The maximum thickness h_0 is then arrived at with the aid of equation (48)

$$h_0 = h(R)\exp(\tfrac{1}{2}\rho\omega^2 R^2/\sigma). \tag{57}$$

6. EXAMINATION OF ERROR ESTIMATES

In the case of the general strain–stress relations, equations (35), (36) and (37), after some manipulation the order of the relative errors in approximations (40), (41) and (45) are

$$O\left[\frac{f'_3(\sigma, \sigma, 0)}{(1/\sigma)f(\sigma, \sigma, 0)} \cdot \frac{\sigma_z}{\sigma}\right] = O\left[\left(\frac{1/E^{\tan}_{\sigma_3}}{1/E^{\sec}_{\sigma}}\right)_{\sigma_3 = 0}\right] O(\tan^2\alpha) \tag{40'}$$

$$O\left[\frac{f'_1(0, \sigma, \sigma)}{(1/\sigma)f(0, \sigma, \sigma)} \cdot \frac{\sigma_z}{\sigma}\right] = O\left[\left(\frac{1/E^{\tan}_{\sigma_1}}{1/E^{\sec}_{\sigma}}\right)_{\sigma_1 = 0}\right] O(\tan^2\alpha) \tag{41'}$$

$$O\left[\frac{\exp(1)}{2} \cdot \frac{g'(0)}{(1/\sigma)f(\sigma, \sigma, 0)} \max(\tan^2\alpha)\right] = O\left[\exp(1)\frac{(1/2G^{\tan})_{\tau = 0}}{(1/E^{\sec}_{\sigma})_{\sigma_3 = 0}}\right] O[\max(\tan^2\alpha)] \tag{45'}$$

The abbreviations $E^{\tan}$ and $E^{\sec}$ denote the "generalized" tangent and secant modulus of elasticity, while $G^{\tan}$ is the tangent shear modulus. The maximum of these error estimates (40'), (41') or (45') reveals the order of the relative error if for the optimum shape there is taken the very simple exponential thickness distribution expressed by equation (48).

Consequently the relative errors are seen to be composed of two factors. The common factor is $\max(\tan^2\alpha)$, due to the simplifications in the principal stresses and the stress trajectories, which can be termed "the geometrical error-factor". Another factor still appears; this has its origin in the elastic properties of the material, and can be termed "the error-factor due to elasticity."

To obtain an idea of the magnitude of errors made in approximations, it is assumed that the material will obey the generalized Hooke's law (cf. equations (35), (36) and (37))

$$\frac{\partial u}{\partial r} = \frac{1}{E}[\sigma_r - \nu(\sigma_\theta + \sigma_z)]$$
$$\frac{u}{r} = \frac{1}{E}[\sigma_\theta - \nu(\sigma_z + \sigma_r)] \tag{35''}$$

$$\frac{\partial w}{\partial z} = \frac{1}{E}[\sigma_z - \nu(\sigma_r + \sigma_\theta)] \tag{36''}$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = \left(\frac{1}{G}\right)\tau_{rz}, \tag{37''}$$

where E and ν are the well-known material constants and $E/2G = 1+\nu$. Equations (40), (41), (42) and (47) give the well-known results

$$e = \frac{u}{r} = (1-\nu)\sigma/E$$
$$\frac{\partial w}{\partial z} = \frac{w}{z} = -2\nu\sigma/E, \tag{47'}$$

while the relative errors are

$$O[|f'_3(\sigma,\sigma,0)\sigma_z/f(\sigma,\sigma,0)|] = O\left(\frac{\nu}{1-\nu}\right)O(\tan^2\alpha_2) \tag{40''}$$

$$O[|f'_1(0,\sigma,\sigma)\sigma_z/f(0,\sigma,\sigma)|] = O\left(\frac{1}{2\nu}\right)O(\tan^2\alpha_2) \tag{41''}$$

$$O[\tfrac{1}{2}\exp(1)g'(0)\sigma\max(\tan^2\alpha)/f(\sigma,\sigma,0)] = O[2{\cdot}72(1+\nu)/(1-\nu)]O[\max(\tan^2\alpha)]. \tag{45''}$$

Since ν is positive, and always less than 0·5, the least possible relative error-factor due to elasticity is of the order $O(3{\cdot}61)$; it is encountered when $\nu \simeq 0{\cdot}14$, which is obtained after equalizing the estimates (41″) and (45″). If $0 < \nu \leq 0{\cdot}14$, the estimate (41″) gives the maximum relative error. However, since for the most usual materials $0{\cdot}14 \leq \nu < 0{\cdot}5$, the maximum relative error is derived from the estimate (45″), and is of the order

$$O(3{\cdot}61\ldots 8{\cdot}16)O[\max(\tan^2\alpha)] \tag{45''}$$

If angle α assumes values from 5 to 10 degrees, the relative error is of the order from 2·8 ... 6·3 to 11·3 ... 25·4 per cent respectively.

7. CONCLUSIONS

The extra condition (that $\sigma_\theta/\sigma_r = s$) leads to $\sigma_\theta = \sigma_r$, and this together with the approximations used has as a direct consequence a constant stress distribution $\sigma_r = \sigma = \sigma_\theta$ throughout. Thus the exponential thickness distribution for the optimum shape is arrived at purely on the basis of the equations of equilibrium. While these equations are completely independent of deformations, and thus of all material properties, it is clear that the stress–strain relations exercise their influence only on the estimates of the relative error, not on the mathematical expression of the optimum shape itself. The relative error is composed of two factors, viz. the geometrical error-factor, and the error-factor due to elasticity. Attention is due to the fact that both error-factors exceed zero in practice, and that the error increases rapidly with the angle α.

It is difficult to determine the moduli of elasticity which appear in the estimates of the relative error in the three-dimensional state of stress for a material that does not obey the generalized Hooke's law. Fortunately, only some of their ratios are needed, and these in turn are presumably weaker functions of stresses. For these reasons the linear law can be taken as a last resort in estimation of the errors. It is accordingly quite safe to accept that the relative error will be of an order which is less than ten times the square of the maximum slope.

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Абстракт—Предлагается новый подход к решению оптимальной формы вращеющегося диска. Он основан, главным образом, на новом начальном предположении /раздел 2/, на приближениях в главных напряжениях и траекториях напряжений /раздел 3/ и на свойствах симметрии зависимостей напряжение-деформация, надлежащих изотропии. Затем он основан также на разложениях рядов Маклорена для этих зависимостей по отношению к σ_2 и τ_{v2} /раздел 4/. Выведенная оптимальная форма оказывается классической. Она уменьщается экспотендиально /раздел 5/ и позваляет оценить относительную погрешность /раздел 6/.